THE CENTRAL WAREHOUSE LOCATION PROBLEM REVISITED*

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Abstract

This paper is concerned with the optimal location of a central warehouse, given a fixed number and the locations of the local warehouses. We investigate whether the solution determined by the traditional model that minimizes total transportation cost differs from the one determined by a model that also takes into account the inventory and service costs. We build simple models to address this question. Numerical results show that ignoring inventory costs in modeling location models may lead to inferior location solutions.

Key Words: Location, Inventory, Central warehouse location.

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1 Introduction

There exists a rich literature in both location theory and inventory theory. These two streams of research, however, are traditionally separated. The research in location problems usually ignores the inventory costs, while the research in inventory problems generally presumes the locations of the warehouses are given. Very little has been written on the integration of the location and inventory decision. Some work has been done in the economics literature associated with the location-production problem. This line of research attempts to characterize production functions for which the optimal location of a firm is independent of the output of the suppliers under various assumptions of transport rates. McCann (1993) applies inventory theory to this framework and shows how the optimal location of a firm depends on the value of materials shipped as well as the production function and transport costs. In the operations research literature, Sigman and Simchi-Levi (1992) give a model to find the optimal location for a mobile server in order to minimize expected customer delay where total inventory level drops by one unit for each completed service. The current paper reflects our desire to combine these two streams of research together. Continuing efforts in this direction, we believe, will provide more insights into effective logistics management.

In particular, we explore the best selection of a central warehouse, given a fixed number and the locations of the local warehouses. Traditionally, in the location research literature, this problem has been approached by minimizing the total transportation costs from the central warehouse to the local warehouses. Under the general notion of reenginneering, however, many companies realized that moving source factories and central warehouses to market areas will not only reduce transportation costs, but also shorten and simplify distribution channels and provide better service with lower inventories (see Harmon 1993). The question is, will the inventory costs and the customer service concerns also affect the optimal central warehouse location? In other words, will the optimal central warehouse location determined by minimizing total transportation cost differ from the one determined by models that also take into account the inventory and service costs?

Obviously, addressing these questions involves joint decision-making: We need to determine the optimal location of the central warehouse and the inventory control policies simultaneously. We can solve the problem sequentially. First, for any given central warehouse location, the problem is a pure inventory problem. Suppose we can find the optimal policy for the inventory problem, then the optimal inventory cost can be expressed as a function of the central warehouse location. The transportation cost is also a function of the central warehouse location. Therefore, the total inventory and transportation costs can be expressed as a function of the central warehouse location. The next step, then, is to optimize this total cost function over all possible central warehouse locations.

In general, it is difficult to express the optimal inventory cost in a closed form. The solutions to most of the inventory models rely on numerical methods, following certain optimality algorithms, rendering the above second step intractable. To overcome this difficulty, in this paper we make certain simplifying but reasonable assumptions on the operation of the central warehouse, so that the optimal inventory cost in the first step has a closed-form solution.

Four models are proposed, analyzed, and solved. In the first model we assume that the backorder cost is known. In the second model we use the common approach that the service level of the system is given and thus the backorder cost (which includes the good will cost which is very difficult to estimate) can be calculated by the given service level. In the third model we assume that maintaining a high service level is more important than minimizing the cost, hence the objective is to maximize the service level. The fourth model, the Weber location problem which ignores inventory costs, is presented for comparison purposes.

The resulting first two models are similar to the well-known Weber problem (Francis et al., 1992; Love et al., 1988) which is termed also the minisum location problem. In particular, the objective is the sum of two weighted sums, one of which is equivalent to the Weber problem and the other is a weighted sum of square roots of some function of the distance metric. The solution procedures for finding the optimal joint decisions and the total cost functions can be done quite efficiently. This facilitates the use of the models to evaluate alternative system designs. For example, one can evaluate whether it is worthwhile for a company to relocate its central warehouse.

The main contribution of this paper is in its innovative models which incorporate inventory costs in a location decision problem. We show that ignoring inventory costs in modeling location models may lead to inferior location solutions.

In Section 2 we present the problem. In Section 3, we derive the total system cost function. In Section 4, we study the structure of the various objective functions and propose algorithms for the solution of the problem. In Section 5, we illustrate some numerical experiments both on an example test problem and randomly generated problems, and perform a sensitivity analysis. Finally, we conclude and summarize the finding of the study in Section 6.

2 Problem Statement

We focus on determining the location of a central warehouse which services n local warehouses located at $P_i = (x_i, y_i)$, $i = 1, 2, \ldots, n$. Demand at warehouse i occurs at a constant rate λ_i. Assume complete backlogging for unsatisfied demands. The service requirement is that the frequency of stockout at warehouse i has to be no greater than $1 - \theta_i$. We assume the central warehouse has ample stock. All the orders from local warehouses are processed simultaneously. Thus, workload congestion at the central warehouse is not a concern. We also assume that only one transportation mode is used, say, by truck. For any given central warehouse location $X = (x, y)$, let $L_i(X)$ be the replenishment lead time for each order placed by local warehouse i. It has two key components: the order processing time τ_i and the transportation time $\beta_i||X - P_i||$ where $||X - P_i|| = ||X - P_i||_p$ is the ℓ_p distance between X and P_i (see Francis et al., 1992 or Love et al. 1988 for the definition and properties of the ℓ_p distance). That is,

$$L_i(X) = \tau_i + \beta_i||X - P_i||.$$

Here, τ_i and β_i are nonnegative constants, and β_i can be interpreted as the inverse of the average velocity of the truck. Similarly, let $k_i(X)$ be the fixed order cost at warehouse i. It also consists of two parts: the administration cost κ_i and the transportation cost $\gamma_i||X - P_i||$. In other words (since each trip is a round trip),

$$k_i(X) = \kappa_i + 2\gamma_i||X - P_i||,$$

where κ_i and γ_i are nonnegative constants, and γ_i is the transportation cost per truck per mile. Other cost factors with respect to local warehouse i are:

$c =$ unit product price at the central warehouse (independent of i).

$v =$ unit shipping cost per mile (in addition to the fixed shipping cost; independent of i;).

$c_i(X) =$ variable ordering cost from warehouse i

$= c + v||X - P_i||$.

$I_i =$ holding cost rate (including interest and opportunity costs, obsolescence and physical depreciation, storage and handling, property taxes, insurance, etc.).

$h_i(X) =$ holding cost rate for each unit of inventory $= I_i c_i(X)$.

$h_0 =$ inventory holding cost rate for each unit of inventory in transit.

$b_i =$ backorder cost rate for each unit of demand backlogged.

3 The Total System Cost

Given the assumptions on the central warehouse operation, and given the central warehouse location X, the inventory problem facing each local warehouse i is a standard EOQ environment: a constant demand rate λ_i, a fixed lead time $L_i(X)$, a fixed ordering cost $k_i(X)$, a variable ordering cost c_i, a holding cost rate $h_i(X)$, and a backorder cost rate b_i. It is shown in the inventory literature (see, e.g., Zipkin (2000), Section 3.3) that the optimal inventory policy for local warehouse i consists a reorder-point $r_i(X)$ and an order quantity $q_i(X)$ which minimizes the average ordering, inventory and backorder cost. In particular, the optimal policy parameters are:

$$\begin{aligned} q_i(X) &= \text{the order quantity } = \sqrt{\frac{2\lambda_i k_i(X)}{h_i(X)\theta_i(X)}} \\ r_i(X) &= \text{the reorder point } = -(1-\theta_i(X))q_i(X) + \lambda_i L_i(X), \end{aligned}$$

where $\theta_i(X) = \frac{b_i}{b_i + h_i(X)}$. With this policy, the stockout proportion is exactly $1 - \theta_i(X)$.

The optimal average ordering and inventory cost at warehouse i is

$$\lambda_i c_i(X) + \sqrt{2\lambda_i k_i(X) h_i(X)\theta_i(X)}.$$

Notice that the variable cost is $c_i(x) = c + v||X - P_i||$. Since constant $c\lambda_i$ is independent of the central warehouse location, we do not need to include it in the total system cost function. That is, the relevant average ordering and inventory cost at warehouse i is

$$v\lambda_i||X - P_i|| + \sqrt{2\lambda_i k_i(X) h_i(X)\theta_i(X)}.$$

On the other hand, the shipments from the central warehouse to a local warehouse are internal to the company. So they incur in-transit holding costs which depend on the central warehouse location. To calculate the average in-transit holding costs, observe that the dynamics of the inventory system at warehouse i repeats in each order cycle (the time between two consecutive orders). Since the demand rate is λ_i and an order of size q_i is placed in each cycle, the cycle length is q_i/λ_i. During each cycle, the stock in transit is q_i in a period of $\beta_i||X-P_i||$ and 0 in the rest of the cycle. Thus, the average stock in transit to the local warehouse i is

$$q_i\frac{\beta_i||X-P_i||}{q_i/\lambda_i}+0\frac{q_i/\lambda_i-\beta_i||X-P_i||}{q_i/\lambda_i}=\lambda_i\beta_i||X-P_i||,$$

which incurs the average stock-in-transit holding cost

$$h_0\lambda_i\beta_i||X-P_i||.$$

To summarize, the total average *system* cost due to warehouse i is

$$\begin{aligned}
TC_i(X) &= \sqrt{2\lambda_i h_i(X)k_i(X)\theta_i(X)}+\lambda_i(v+h_0\beta_i)||X-P_i|| \\
&= \sqrt{2\lambda_i I_i(c+v||X-P_i||)(\kappa_i+2\gamma_i||X-P_i||)\theta_i(X)}+\lambda_i(v+h_0\beta_i)||X-P_i|| \\
&= \sqrt{2\lambda_i I_i\theta_i(X)}\sqrt{c\kappa_i+(2c\gamma_i+v\kappa_i)||X-P_i||+2v\gamma_i||X-P_i||^2}+\lambda_i(v+h_0\beta_i)||X-P_i||.
\end{aligned}$$

The total average system cost of the firm is:

$$TC(X)=\sum_{i=1}^{n}TC_i(X).$$

Thus, the optimization problem of interest can be written as:

$$\min\left\{\sum_{i=1}^{n}\sqrt{2\lambda_i I_i\theta_i(X)}\sqrt{c\kappa_i+(2c\gamma_i+v\kappa_i)||X-P_i||+2v\gamma_i||X-P_i||^2}+\sum_{i=1}^{n}\lambda_i(v+h_0\beta_i)||X-P_i||\right\}. \tag{1}$$

We distinguish between two cases which lead to two different models. In the first model we take the practical approach that since it may be difficult to estimate the value of the backorder cost because part of it is goodwill cost, that $\theta_i(X)$ is fixed by the manager. In such a case b_i becomes dependent on X, $\theta_i(X)$ becomes independent of X (and given). In the second model we assume that b_i is given and thus $\theta_i(X)$ is a function of X.

3.1 Model 1: Proportional backorder cost rate

One common case is that the backorder cost rate b_i is proportional to the holding cost rate h_i, i.e., $b_i = mh_i$ for some positive number m. This is reasonable if b_i is proportional to the unit cost of the product. For example, if b_i constitutes mostly the lost revenue, then $b_i = c_i$. (We already assume h_i is proportional to c_i.) It follows that $\theta_i = \frac{m}{m+1}$ which is independent of X. For simplicity, we set

$$w_i = \sqrt{2\lambda_i I_i \theta_i}, \quad \alpha_i = \lambda_i(v + h_0\beta_i), \quad A_i = 2v\gamma_i, \quad B_i = 2c\gamma_i + v\kappa_i, \quad C_i = c\kappa_i. \tag{2}$$

Hence, the problem can be expressed as

$$\min \left\{ TC_1(X) = \sum_{i=1}^{n} \left(\alpha_i ||X - P_i|| + w_i \sqrt{A_i ||X - P_i||^2 + B_i ||X - P_i|| + C_i} \right) \right\}. \tag{3}$$

3.2 Model 2: Given backorder cost rate

Since $\theta_i(X) = \frac{b_i}{b_i + I_i c + I_i v ||X - P_i||}$, Equation (1) is written as:

$$\min \left\{ \sum_{i=1}^{n} \sqrt{2\lambda_i I_i b_i} \sqrt{\frac{c\kappa_i + (2c\gamma_i + v\kappa_i)||X - P_i|| + 2v\gamma_i ||X - P_i||^2}{b_i + I_i c + I_i v ||X - P_i||}} + \sum_{i=1}^{n} \lambda_i (v + h_0\beta_i) ||X - P_i|| \right\}. \tag{4}$$

For simplicity, we add the following notations:

$$v_i = \sqrt{2\lambda_i I_i \frac{b_i}{b_i + I_i c}}, G_i = \frac{I_i v}{b_i + I_i c}. \tag{5}$$

Hence, the problem can be expressed as

$$\min \left\{ TC_2(X) = \sum_{i=1}^{n} \left(\alpha_i ||X - P_i|| + v_i \sqrt{\frac{A_i ||X - P_i||^2 + B_i ||X - P_i|| + C_i}{1 + G_i ||X - P_i||}} \right) \right\}. \tag{6}$$

3.3 Model 3: The Maximin Model

It may be more appropriate to provide the best possible service rather minimizing total cost. This means to maximize the level of service. Our objective is therefore to:

$$\max_X \left\{ \min_i \{\theta_i(X)\} \right\} \tag{7}$$

Since $\theta_i(X) = \frac{b_i}{b_i + I_i c + I_i v ||X - P_i||}$, Equation (7) is equivalent to:

$$\min_X \left\{ H(X) = \max_i \{ \frac{I_i c}{b_i} + \frac{I_i v}{b_i} ||X - P_i|| \} \right\} \tag{8}$$

3.4 Model 4: The Weber Model

For the purpose of comparison, we will also consider the following Weber model:

$$\min \left\{ C^W(X) = \sum_{i=1}^{n} \lambda_i ||X - P_i|| \right\}. \tag{9}$$

Clearly, the Weber model does not take into account the costs associated with inventories. Its objective function is linear in the distance between the central warehouse and the local warehouses, and the weights are just the demand rates at the local warehouses, whereas in (3) and (6) such dependencies are more intricate.

4 Analysis

4.1 Analysis of Model 1

We first establish some properties of $TC_1(X)$ in Equation (3).

Lemma 1: *Let X and X' be two locations. If $||X - P_i|| \leq ||X' - P_i||$ $\forall i$, then $TC_1(X) \leq TC_1(X')$.*

Proof: Self evident. □

Theorem 1: *An optimal solution exists in the convex hull of the local warehouses.*

Proof: Consider a point X' outside the convex hull. There exists a point X in the convex hull which is not farther from all demand points. By Lemma 1 $TC_1(X) \leq TC_1(X')$, hence, X' cannot be the only optimal solution. $\square$

The function $TC_1(X)$ in Equation (3) is not generally a convex function. We prove that it is a sum of quasi-convex functions.

Lemma 2: *The function $y = \sqrt{ax^2 + bx + c}$ $(a,\ b,\ c \geq 0)$ is concave if $b^2 \geq 4ac$.*

Proof: By direct derivation:

$$y\prime = \frac{2ax + b}{2\sqrt{ax^2 + bx + c}},$$

$$y\prime\prime = \frac{4ac - b^2}{4\left[\sqrt{ax^2 + bx + c}\right]^3} \leq 0.$$

$\square$

Lemma 3: *$B_i^2 \geq 4A_iC_i$ where A_i, B_i, C_i were defined in Equation (2).*

Proof: Since the arithmetic mean is greater than or equal to the geometric mean, $(2c\gamma_i + \upsilon\kappa_i)^2 \geq 8c\gamma_i\upsilon\kappa_i$. By direct substitution:

$$B_i^2 = (2c\gamma_i + \upsilon\kappa_i)^2 \geq 8c\gamma_i\upsilon\kappa_i \geq 4A_iC_i$$

$\square$

Although convexity cannot be proven in general, each individual term of $TC_1(X)$ is quasi-convex. This is proven in the next Theorem:

Theorem 2: *The individual terms of $TC_1(X)$ in Equation(3) are quasiconvex.*

Proof: The term

$$\alpha_i||X - P_i|| + w_i\sqrt{A_i||X - P_i||^2 + B_i||X - P_i|| + C_i}$$

is monotonically increasing in the distance $||X - P_i||$ by Lemma 1. The distance function is convex (Francis et al., 1992; Love et al., 1988). An increasing function of a convex function is quasiconvex (Avriel et al., 1988) which proves the Theorem. □

Hence the objective function $TC_1(X)$ being a sum of quasiconvex functions is not expected to be quasiconvex (or convex). Our computational experiments bear this out and we may safely conclude that $TC_1(X)$ is nonconvex.

One can solve (3) using standard general purpose optimization software such as AMPL (Fourer et al. 1993), or the solver in Excel. We propose a specially designed Weiszfeld-type procedure for the Euclidean distance case. Such an algorithm finds a local minimum of problem (3). The principle is explained in the book chapter by Drezner and Drezner (1998). Suppose the optimization problem is:

$$\min\left\{f(x,y) = \sum_{i=1}^{n} f_i(d_i(x,y))\right\},$$

where $f_i(d_i)$ are functions of the Euclidean distance $d_i(x,y)$ to local warehouse i. By direct differentiation, the conditions for a local minimum are:

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{n}\left\{\frac{1}{d_i(x,y)}\frac{\partial f_i}{\partial d_i}(x - x_i)\right\} = 0,$$

$$\frac{\partial f}{\partial y} = \sum_{i=1}^{n}\left\{\frac{1}{d_i(x,y)}\frac{\partial f_i}{\partial d_i}(y - y_i)\right\} = 0.$$

Solving these equations implicitly leads to the following recursive relationship:

$$x = \frac{\sum_{i=1}^{n}\left\{\frac{1}{d_i(x,y)}\frac{\partial f_i}{\partial d_i}x_i\right\}}{\sum_{i=1}^{n}\left\{\frac{1}{d_i(x,y)}\frac{\partial f_i}{\partial d_i}\right\}}$$

$$y = \frac{\sum_{i=1}^{n}\left\{\frac{1}{d_i(x,y)}\frac{\partial f_i}{\partial d_i}y_i\right\}}{\sum_{i=1}^{n}\left\{\frac{1}{d_i(x,y)}\frac{\partial f_i}{\partial d_i}\right\}}$$

For the problem of minimizing $TC_1(X)$, $f_i(d_i) = \alpha_i d_i + w_i\sqrt{A_i d_i^2 + B_i d_i + C_i}$. Therefore:

$$\frac{\partial f_i}{\partial d_i} = \alpha_i + w_i \frac{2A_i d_i + B_i}{2\sqrt{A_i d_i^2 + B_i d_i + C_i}}.$$

This leads to the recursive relationship:

$$\begin{aligned} x &= \frac{\sum_{i=1}^{n}\left\{\frac{1}{||X-P_i||}\left[\alpha_i + w_i\frac{2A_i||X-P_i||+B_i}{2\sqrt{A_i||X-P_i||^2+B_i||X-P_i||+C_i}}\right]x_i\right\}}{\sum_{i=1}^{n}\left\{\frac{1}{||X-P_i||}\left[\alpha_i + w_i\frac{2A_i||X-P_i||+B_i}{2\sqrt{A_i||X-P_i||^2+B_i||X-P_i||+C_i}}\right]\right\}}, \\ y &= \frac{\sum_{i=1}^{n}\left\{\frac{1}{||X-P_i||}\left[\alpha_i + w_i\frac{2A_i||X-P_i||+B_i}{2\sqrt{A_i||X-P_i||^2+B_i||X-P_i||+C_i}}\right]y_i\right\}}{\sum_{i=1}^{n}\left\{\frac{1}{||X-P_i||}\left[\alpha_i + w_i\frac{2A_i||X-P_i||+B_i}{2\sqrt{A_i||X-P_i||^2+B_i||X-P_i||+C_i}}\right]\right\}}. \end{aligned} \tag{10}$$

The Weiszfeld-like procedure starts with a point in the plane. Each iteration, the current point is substituted in the right hand side of Equation (10) and the resulting left hand side is the location for the next iteration. The procedure is stopped when the distance between the locations of two successive iterations is below a pre-specified tolerance. Such a procedure has been shown to converge for the Weber problem (i.e., $w_i = 0$ for our problem) (Kuhn, 1967, Ostrech, 1978). Since the terms in the denominator are all positive, the procedure is expected to behave similarly to the Weiszfeld procedure.

Morris (1991) showed, for a Weber problem with different distances (generalized ℓ_p), that a Weiszfeld procedure converges. An essential property for his proof, that also holds for our Weiszfeld procedure, is that every iteration of the algorithm is in the convex hull of the local warehouses because it is a weighted average (with positive weights) of the locations of the local warehouses.

Therefore, the sequence of locations is in the convex hull which is a compact set. Invoking the Bolzano-Weierstrass Theorem, we know that the sequence of locations has at least one concentration point. However, since our problem is expected to be non-convex, there might be more than one local minimum which suggests that in some cases the global optimum may not be obtained.

4.2 Analysis of Model 2

We prove that the optimal solution to problem (6) is also in the convex hull of the local warehouses. First, we prove that $TC_2(X)$ is a monotonically increasing function of the distance.

Lemma 4: *Let X and X' be two locations. If $||X - P_i|| \leq ||X' - P_i||\ \forall i$, then $TC_2(X) \leq TC_2(X')$.*

Proof: Consider the function y which represents the second individual term in equation (6) by the following relationship:

$$y^2(1 + gx) = ax^2 + bx + c$$

by finding the derivative of this expression

$$2yy'(1 + gx) = 2ax + b - gy^2 = \frac{agx^2 + 2ax + b - gc}{1 + gx} \tag{11}$$

To show that $y' \geq 0$, we need to show that

$$agx^2 + 2ax + b - gc \geq 0.$$

To show this we show that $b - gc \geq 0$ for equation (6). Indeed,

$$B_i - G_iC_i = 2c\gamma_i + v\kappa_i - \frac{I_iv}{b_i + I_ic}c\kappa_i \geq 2c\gamma_i + v\kappa_i - \frac{I_iv}{I_ic}c\kappa_i = 2c\gamma_i \geq 0$$

□

Note that Lemma 4 is only true for the special definitions of the coefficients in the equation. It is not generally true for any non-negative A_i, B_i, C_i, and G_i.

Theorem 3: *An optimal solution exists in the convex hull of the local warehouses.*

Proof: Identical to the proof of Theorem 1. $\square$

A sufficient condition for the convexity of $TC_2(X)$ can be obtained by solving the equation $y'' \geq 0$. However, the condition depends on the distance, and is too tedious to present in this paper. However, quasiconvexity can be proved for the individual terms of $TC_2(X)$ in Equation(6).

Theorem 4: *The individual terms of $TC_2(X)$ in Equation(6) are quasiconvex.*

Proof: The term

$$\alpha_i||X - P_i|| + v_i\sqrt{\frac{A_i||X - P_i||^2 + B_i||X - P_i|| + C_i}{1 + G_i||X - P_i||}}$$

is monotonically increasing in the distance $||X - P_i||$ by Lemma 4. An increasing function of a convex function is quasiconvex (Avriel et al., 1988) which proves the Theorem. $\square$

A generalized Weiszfeld algorithm can be constructed for Model 2 as well. For this case, by equation (11):

$$\frac{\partial f_i}{\partial d_i} = \alpha_i + v_i\frac{A_iG_id_i^2 + 2A_id_i + B_i - G_iC_i}{2(1 + G_id_i)^{1.5}\sqrt{A_id_i^2 + B_id_i + C_i}}$$

This leads to a Weiszfeld procedure with positive weights (by the proof of Lemma 4 $B_i - G_iC_i \geq 0$) which also guarantees that the iterations are all in the convex hull of the local warehouses. For terseness we do not spell out the Weiszfeld formula.

4.3 Analysis of Model 3

This problem is the weighted minimax location problem with a set-up cost. It is analyzed in Drezner (1991) where an iterative solution procedure is proposed. The problem is convex hence a local minimum is a global one. Once the objective function is converted to the θ's, it is a maximization of a concave function.

The iterative procedure is similar to the one proposed in Elzinga and Hearn (1972). It is based on the observation that there is a set of up to three local warehouses such that the optimal solution to this partial set is the optimal solution for the problem based on the complete set of local warehouses. We start by randomly selecting a starting solution. The three local warehouses with the highest value of the objective function are selected, and the solution to this set of three warehouses found. If there is a local warehouse whose value of the objective function is greater than the maximum for the three chosen local warehouses, it replaces one of the three selected local warehouses, and the new problem solved. The process continues until the selected group of three local warehouses is the solution to the complete problem. This procedure is very efficient and in practice requires very few iterations.

4.4 Analysis of Model 4

The Weber location problem is extensively analyzed and many methods exist for its solution (Francis et al., 1992; Love et al., 1988). The objective function is convex and thus a local minimum is the global one. The iterative Weiszfeld procedure (Weiszfeld, 1936) is the recommended approach. Drezner (1996) proposed an accelerated approach which cuts the run time by about one half.

5 Computational Experience

In this section we perform three types of experiments. We solve a small illustrative example problem using Excel, perform a sensitivity analysis on the parameters of this problem, and solve large randomly generated problems using the generalized Weiszfeld algorithm.

Table 1: Characteristics of the local warehouses for the example problem

i	x_i	y_i	λ_i
1	0	0	0.33
2	4000	4000	0.13
3	2000	500	0.02
4	0	4000	0.22
5	500	2000	0.06
6	4000	0	0.63

5.1 The Example Problem

An example problem containing 6 local warehouses was solved by using the Solver in Excel. The parameters for the problem are given in Table 1. All other parameters were the same for every local warehouse. They were: $\beta_i = 0.0005$, $\kappa_i = 50$, $\gamma_i = 0.4$, $I_i = 0.3$, $b_i = 30$, $h_0 = 0.003$, $v = 0.01$, $c = 30$, and $\theta = 0.95$. Euclidean distances were used. In Figure 1 we depict the local warehouses and the four locations obtained by the four different models. In Table 2 we depict the value of the objective function by a model at the optimal location for another model. The optimal value of the objective function for each model is denoted in boldface. For example, the optimal location for model 1 is at (2000, 500) with an objective function of 574.61. If one would apply model 4 and locate the warehouse at (3788, 120), which is the optimal location by model 4, the cost would be 589.56 which is 2.6% higher. If one will use model 3 for selecting the location for the warehouse, its objective function would be 593.87 or 3.4% higher.

In Figure 2 the three-dimensional surface of the value of the objective function for the four models is depicted. It is clear from the figure that models 1 and 2 are not convex. Model 3 is clearly concave with a unique local maximum which is the global one. Model 4 is clearly convex with a unique local minimum which is the global one.

Table 2: Solutions to the example problem

Model	Solution		Objective by Model			
	x	y	1	2	3(θ)	4
1	2000	500	**574.61**	505.19	0.704	3517.3
2	4000	0	577.00	**500.42**	0.680	3367.6
3	2000	2000	593.87	523.44	**0.722**	3825.2
4	3788	120	589.56	513.19	0.684	**3366.0**

5.2 Sensitivity Analysis

We investigated the sensitivity of the solutions to the example problem by the four models to the parameters of the example problem. In Table 3 we report results for Models 1 and 2. Since the solutions for Models 1 and 2 are at local warehouses, we calculated the ranges for each parameter for which the solution remains at the same location. Within these ranges the value of the parameter does not affect the solution. Note that b is not a parameter of Model 1 and θ is not a parameter of Model 2 and thus are denoted by NA (not applicable).

By examining Table 3 we observe that most parameters have quite a wide range that leads to the same solutions for model 1 and model 2. The parameter λ_3 has the tightest range. This means that the solution is most sensitive to the demand rate at local warehouse #3.

The solution to Model 3 is not sensitive to the parameters of the problems as long as the same holding cost ratio (I_i) and backorder cost (b_i) are used for all local warehouses. In this case, Model 3 solution is at the center of the circle enclosing all local warehouses regardless of the parameters' values. Model 4 is the standard Weber problem and presented only for comparison.

Table 3: Ranges for the parameters for which the solution remains the same

Variable Name	Parameter	Base Value	Model 1 Lower Bound	Model 1 Upper Bound	Model 2 Lower Bound	Model 2 Upper Bound
Speed inverse	β	0.0005	0	5.3	0	∞
Administration cost	κ	50	35	∞	0	97
Transportation cost	γ	0.4	0	0.55	0.2	∞
Holding cost ratio	I	0.3	0.05	∞	0	∞
Holding cost in transit	h_0	0.003	0.0027	∞	0	∞
Shipping cost per mile	v	0.01	0	0.025	0	∞
Product price	c	30	0	34	23	∞
Backorder cost	b	30	NA	NA	0	125
Service level	θ	0.95	0.15	1	NA	NA
Demand Rates	λ_1	0.33	0.31	0.59	0	0.38
	λ_2	0.13	0	0.75	0	0.74
	λ_3	0.02	0.017	∞	0	0.030
	λ_4	0.22	0.20	0.42	0	0.28
	λ_5	0.06	0.05	0.13	0	0.09
	λ_6	0.63	0.39	0.65	0.59	∞

5.3 Randomly Generated Problems

The example problem is not convex for models 1 and 2. In order to assess whether these models are usually non-convex, we solved randomly generated problems using the generalized Weiszfeld algorithms. A second purpose of these tests is to evaluate the efficiency of solving larger problems. Programs in Fortran PowerStation 4.0 were coded and ran on a 600MHz Pentium III Toshiba Portege 7200 laptop computer. We ran problems with the number of local warehouses ranging from 5 to 10,000. The parameters for the problems were randomly generated as follows: the locations of the local warehouses were uniformly generated in a square of side 4000, λ_i were uniformly generated in [0, 1], and the other parameters (except I_i and θ_i which were kept the same) were uniformly generated between the values used for the example problems and five times these values. Each problem was solved 1000 times starting from randomly generated starting solutions. The results

Table 4: Solving randomly generated problems using the generalized Weiszfeld algorithm

	Model 1			Model 2		
n	†	‡	Time§	†	‡	Time§
5	3	994	0.04	3	990	0.08
10	1	1000	0.12	5	984	0.21
50	1	1000	0.72	1	1000	1.72
100	1	1000	1.47	2	952	18.24
500	1	1000	6.49	1	1000	19.08
1000	1	1000	13.49	1	1000	46.05
5000	1	1000	67.43	1	1000	280.16
10000	1	1000	132.80	1	1000	490.34

† Number of different local minima found

‡ Number of times that best local minimum found

§ Total time in seconds for all 1000 runs

are depicted in Table 4.

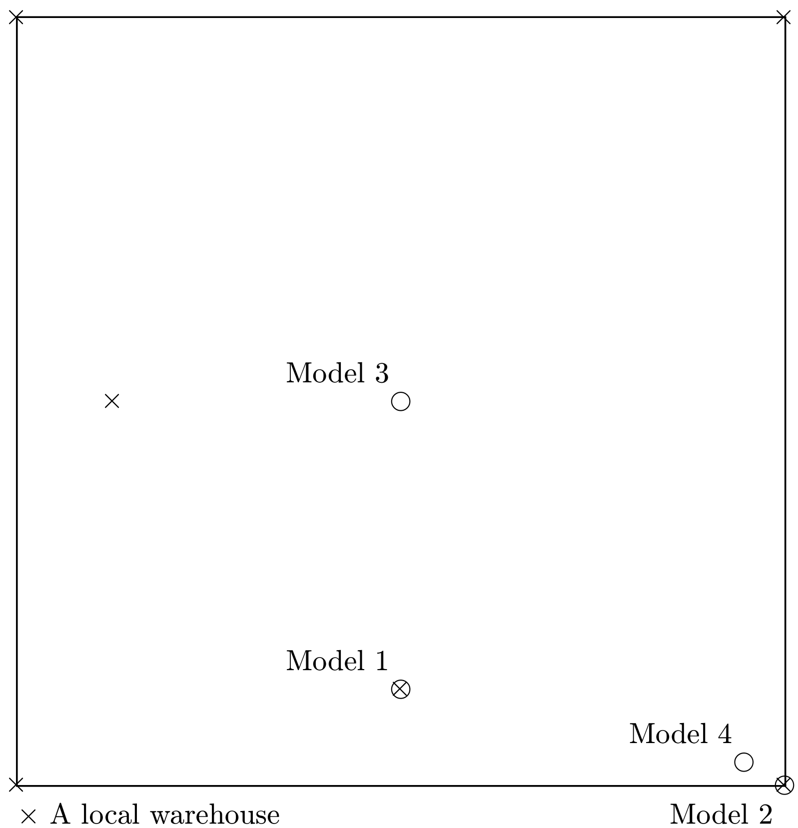
As can be seen from the Table, in most cases only one local minimum was identified. Only one problem ($n = 5$) found more than one local minimum for Model 1. Three cases resulted in more than one local minimum for Model 2. In all cases, at least 95% of the runs resulted in the best found local minimum. The generalized Weiszfeld algorithm is very efficient. It took less than half a second to solve a problem with 10,000 local warehouses.

6 Conclusions

There are many models analyzing the inventory aspect in supply chain management. A different body of the literature is concerned with minimizing the transportation cost in supply chain management by selecting the best location for a warehouse. In this paper we unify both approaches into one by minimizing the total transportation and inventory cost by selecting the best location for a warehouse.

Very efficient methods are proposed for the solution of these models. We solved problems of up

Figure 1: Solutions for a test problem



× A local warehouse

○ A solution by a particular model

to 10,000 demand points in less than half a second. We demonstrate the models on an example problem which is solved by the Solver in Excel. It turns out that the location solutions for the four models are quite different from one another. We conclude that different models lead to different locations and thus the decision maker needs to decide which model is the most appropriate for the situation at hand.

As future research we propose to extend the problem to the global situation by using spherical distances, investigate models with stochastic demand, model and solve the multiple central warehouse problem, and possibly have a capacity constraint on the central warehouses.

References

1. Avriel M., W.D. Diewert, S. Schaible, and I. Zang (1988) *Generalized Concavity (Mathematical Concepts and methods in Science and engineering, Vol. 36)*, Plenum Publishing.
2. Drezner Z. (1991) "The Weighted Minimax Location Problem with Set-up Costs and Extensions," *RAIRO - Operations Research*, 25, 55-64.
3. Drezner Z. (1996) "A Note on Accelerating the Weiszfeld Procedure," *Location Science*, 3, 275-279.
4. Drezner Z. and T. Drezner (1998) "Applied Location Theory Models," in *Modern Methods for Business Research*, G.A. Marcoulides (Ed.), Lawrence Erlbaum Associates, Mahwah, NJ.
5. Elzinga, D.J. and D.W. Hearn (1972) "Geometrical Solutions for Some Minimax Location Problems," *Transportation Science*, 6, 379-394.
6. Fourer, R., D.M. Gay and B.W. Kernighan (1993) "AMPL a Modelling Language for Mathematical Programming," *the Scientific Press*, South San Francisco.
7. Francis R.L., L.F. McGinnis Jr., and J.A. White (1992) *Facility Layout and Location: An Analytical Approach*, 2nd Edition, International Series in Industrial and Systems Engineering, Prentice Hall, Englewood Cliffs, NJ.
8. Harmon, R. (1993) *Reinventing the Warehouse,* The Free Press. New York, NY.
9. Hadley, G. and T. Whitin. (1963). *Analysis of Inventory Systems,* Prentice Hall, Englewood Cliffs, NJ.
10. Kuhn, H.W. (1967) "On a Pair of Dual Nonlinear Programs," *Methods of Nonlinear Programming*, J. Abadie, Editor, North Holland, Amsterdam, Netherlands.

11. Love, R.F., J.G. Morris and G.O. Wesolowsky (1988) *Facilities Location: Models and Methods*, North Holland, NY.

12. McCann, P. (1993). "The Logistics-Cost Location-Production Problem," *Journal of Regional Science*, **33**, 503-516.

13. Morris J.G. (1981). "Convergence of the Weiszfeld Algorithm for weber Problems Using a generalized "Distance" Function," *Operations Research*, 29, 37-48.

14. Ostresh, L.M. Jr. (1978) "On the Convergence of a Class of Iterative Methods for Solving the Weber Location Problem," *Operations Research*, 26, 597-609.

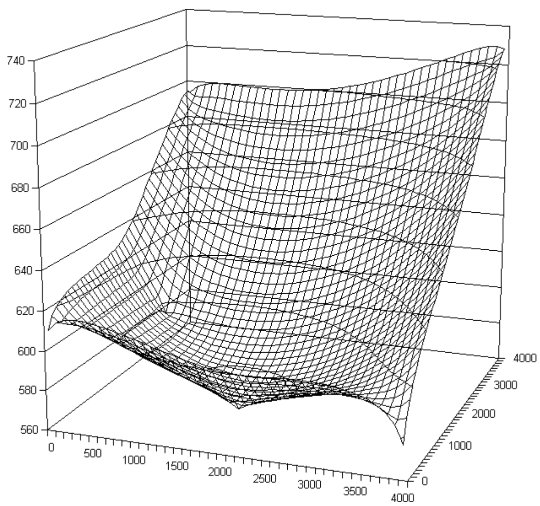
15. Sigman, K. and D. Simchi-Levi (1992). "Light Traffic Heuristic for an M/G/1 Queue with Limited Inventory," *Annals of Operations Research*, 40, 371-380.

16. Weiszfeld, E. (1936) "Sur Le Point Pour Lequel La Somme Des Distances De N Points Donnés Est Minimum," *Tohoku Mathematical Journal*, 43, 355-386.

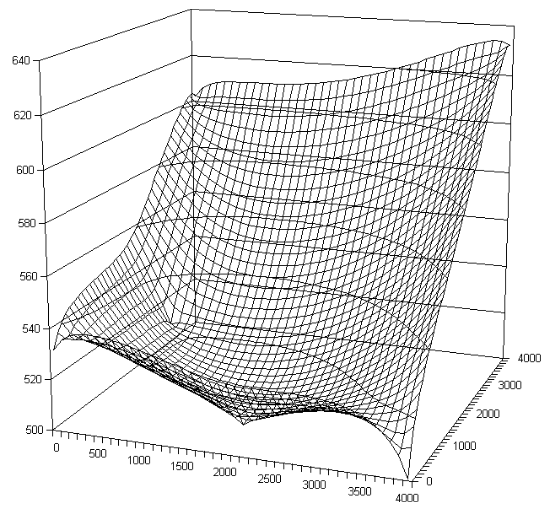
17. Zipkin, P., (2000) *Foundations of Inventory Management,* McGraw-Hill, New York.

Figure 2: The Objective Function of the Four Models for the Example Problem

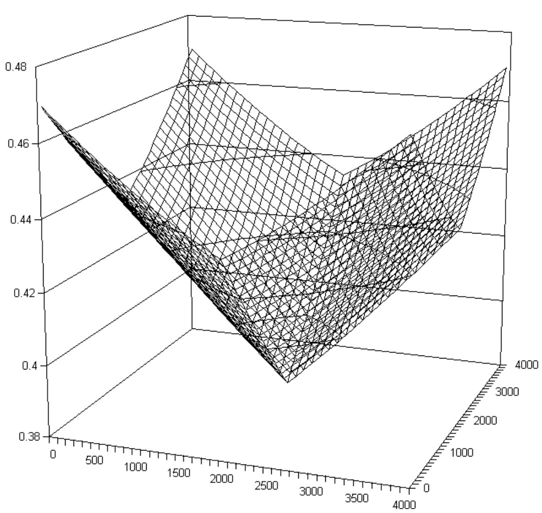
Model 1



Model 2



Model 3



Model 4